Continued /Unit - 1

[1] **Sine Formula** :

To prove that the sines of the angle s of a spherical triangle are proportional to the sines of opposite sides,

$$\frac{SinA}{Sina} = \frac{SinB}{Sinb} = \frac{SinC}{Sinc}$$

Proof: Since we know that

$$\text{CosA} = \frac{Cosa - Cosb\ Cosc}{Sinb\ Sinc}$$

Then, $Sin^2A = 1 - \ Cos^2A = 1 - \left(\frac{Cosa - Cosb\ Cosc}{sinb\ sinc}\right)^2$

$$= \frac{\text{Sin}^2\text{b Sin}^2\text{c} - (\text{Cosa} - \text{Cosb Cosc})^2}{\text{Sin}^2\text{b Sin}^2\text{c}}$$

$$= \frac{(1 - \text{Cos}^2\text{ b})\ (1 - \text{Cos}^2\text{c}) - (\text{Cosa} - \text{Cosb Cosc})^2}{\text{Sin}^2\text{b Sin}^2\text{c}}$$

$$= \frac{-Cos^2b\ \ Cos^2c + 2\ Cosa\ Cosb\ Cosc + 1 - Cos^2b - Cos^2\ c + Cos^2\ bCos^2c - Cos^2\ a}{Sin^2b\ Sin^2c}$$

$$\text{SinA} = \frac{\left(1 - Cos^2a - Cos^2b - Cos^2c + 2Cosa\ Cosb\ Cosc\right)^{1/2}}{Sib\ Sinc}$$

We have taken positive sign only with the radical (square root) because the angles and sides of a spherical triangle are each less than two-right angles and so sinb , sinc and SinA are all positive. Thus,

$$\frac{Sin\ A}{Sina} = \frac{\left(1 - cos^2a - cos^2b - cos^2c + 2\ cosa\ cosb\ cosc\right)^{1/2}}{Sinb\ Sinc}$$

Similarly we can write

$$\frac{SinA}{Sina} = \frac{SinB}{Sinb} = \frac{SinC}{Sinc} = \frac{\left(1 - Cos^2a - Cos^2b - Cos^2c + 2\ Cosa\ Cosb\ Cosc\ \right)^{1/2}}{Sina\ Sinb\ Sinc}$$

$=\frac{2n}{Sina\ Sinb\ Sinc}$ where $4n^2 = 1 - Cos^2a - Cos^2b - Cos^2c + 2Cosa\ Cosb\ Cosc$

$$4n^2 = \begin{vmatrix} 1 & Cosc & Cosb \\ Cosc & 1 & Cosa \\ Cosb & Cosa & 1 \end{vmatrix}$$

Therefore, $\frac{SinA}{Sina} = \frac{SinB}{Sinb} = \frac{SinC}{Sinc} = \frac{2n}{Sina\ Sinb\ Sinc}$ hence proved.

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[1.1] *Formulae for Half an angle*:

In a spherical triangle ABC, show that,

(i) $\text{Sin}\frac{A}{2} = \sqrt{\frac{Sin\ (s-b) Sin\ (s-c)}{Sinb\ Sin\ c}}$;

(ii) $\text{Cos}\frac{A}{2} = \sqrt{\frac{SinS\ Sin\ (s-a)}{Sinb\ Sin\ c}}$;

(iii) $\tan\frac{A}{2} = \sqrt{\frac{Sin(s-b)\ Sin\ (s-c)}{Sins\ Sin\ (s-c)}}$

(iv) $\text{SinA} = \frac{2n}{Sinb\ Sinc}$; $n^2 = Sins\ Sin(s-a) Sin(s-b) Sin(s-c)$
and 2s= (a + b + c)

Proof: We know that

$$\textbf{CosA} = \frac{\boldsymbol{Cosa - Cosb\ Cosc}}{\boldsymbol{Sinb\ Sinc}}$$

$1 - CosA = 1 - \frac{Cosa - Cosb\ Cosc}{Sinb\ Sinc}$, substracting each term from 1 both sides

$$2Sin^2\ \frac{A}{2} = \frac{Sinb\ Sinc - Cosa + Cosb\ Cosc}{sinb\ sinc}$$

$$= \frac{Cosb\ Cosc + Sinb\ Sinc - Cosa}{Sinb\ Sinc}$$

$$= \frac{(Cosb\ Cosc + Sinb\ Sinc\ ) - (Cosa)}{Sinb\ Sin\ c}$$

$= \frac{Cos\ (b-c) - Cosa}{Sinb\ Sinc}$ since, (CosC-CosD= 2Sin (C+D)/2 Sin (D-C)/2)

$$\frac{2Sin\left(\frac{a+b-c}{2}\right) Sin\left(\frac{a-b+c}{2}\right)}{Sinb\ Sin\ c}$$

If 2s = a+b+c then, a+b-c =a+b+c -2c= 2 (s-c)

And a-b+c= a+b+c-2b= 2s-2b=2(s-b)

$$Sin^2\frac{A}{2}= \frac{Sin\ (s-b)Sin\ (s-c)}{Sinb\ Sinc}$$

$$Sin\frac{A}{2} = \sqrt{\frac{Sin(s-b)Sin\ (s-c)}{Sin\ b\ Sinc}} \qquad (1)$$

The positive sign before the radical (i.e Square root) is taken since A is less than two right angles or (A/2) is less than one right angle and so sin (a?2), Cos (A/2) , tan (A/2) are all positive.

[ii] Again we have 1+cosA= $1+\frac{Cosa-Cosb\ Cosc}{Sinb\ Sinc}$

Or $2Cos^2\frac{A}{2}= \frac{Sinb\ Sinc+Cosa-Sinb\ Sinc}{Sinb\ sinc}$

Or$2Cos^2\frac{A}{2}= \frac{Cosa-(Cosb\ Cosc-Sinb\ Sinc)}{Sinb\ Sinc}$

Or $2Cos^2\frac{A}{2}= \frac{Cosa-Cos(b+c)}{sinb\ sinc}$

Or $2Cos^2\frac{A}{2}= \frac{2\sin\frac{a+b+c}{2}\ Sin\left(\frac{b+c-a}{2}\right)}{Sinb\ Sinc}$

Or $Cos^2\frac{A}{2}= \frac{Sins\ Sin\ (s-a)}{Sinb\ Sinc}$

Or $Cos\frac{A}{2} = \sqrt{\frac{Sins\ Sin\ (s-a)}{Sin\ b\ Sin\ c}}$ (2)

Dividing (1) by (2) we get, $tan\frac{A}{2} = \sqrt{\frac{Sin\ (s-b)Sin\ (s-c)}{Sins\ Sin\ (s-a)}}$ (3)

Which proves (iii)

Again, SinA= $2Sin\frac{A}{2}Cos\ \frac{A}{2}$ then we have,

SinA= $\frac{2}{Sinb\ Sinc}[Sins\ Sin(s-a)Sin(s-b)Sin(s-c)]^{1/2}$

From (1) and (2)

Or $SinA = \frac{2n}{Sinb\ Sinc}$ (4)

Where n^2= SinS Sin (s-a) Sin(s-b) Sin (s-c)

Where *n* is called the Norm of the sides of the spherical triangle ABC.

From above we conclude that ,

$$\frac{SinA}{Sina} = \frac{SinB}{Sinb} = \frac{SinC}{Sinc} = \frac{2n}{Sina\ Sinb\ Sinc}$$

(**Maj. 2017 - G.U)**

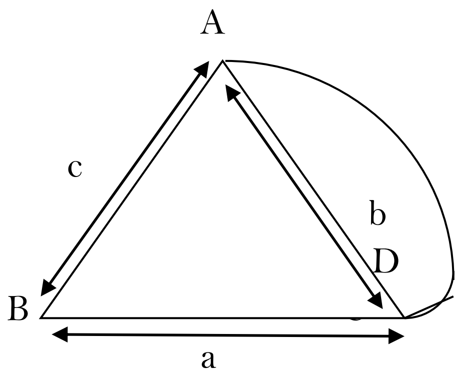
HOME WORK

(a) $Sin\frac{a}{2} = \sqrt{\frac{-CosS\ Cos\ (S-A)}{SinB\ SinC}}$

$(b)Cos\frac{a}{2} = \sqrt{\frac{Cos\ (S-B)Cos\ (S-C)}{SinB\ Sin\ C}}$ and (c) $\tan\frac{a}{2} = \sqrt{\frac{-CosS\ Cos\ (S-A)}{Cos\ (S-B)Cos\ (s-C)}}$;

$$(d)Sina = \frac{2N}{SinB\ SinC}$$

[2] **Sine-Cosine Formula** *: Relation between three (3) sides and two (2) angles of a spherical triangle.*



Let ABC be any spherical triangle, and produce side BC of the triangle to D so that BD= 90^0. Then CD= (90^0-a) and $\angle ACD = 180^0 - C$ join A and D by great circle arc.

From ΔADB *by cosine formula we have*, $CosAD = Cosc\ Cos90^0 + Sinc\ Sin90^0 CosB$

Or $CosAD = Sinc\ CosB$ (1)

Also from ΔADC *we have by Cosine formula*

Or

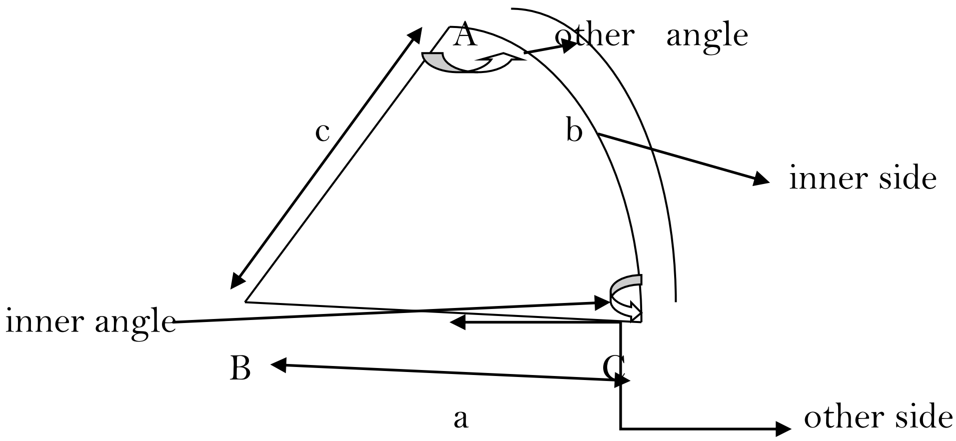
$$CosAD = Cosb\ Cos\ (90^0 - a) + Sinb\ Sin(90^0 - a)Cos\ (180^0 - C)$$

Or $CosAD = Sina\ Cosb - Cosa\ Sinb\ CosC$ (2)

From Eqs.(10 and (2) we have,

Sinc CosB = Sina Cosb-Cosa Sinb CosC, which is the required formula. Proved.

[3] **Cotangent (i.e Cot) Formula** : *Relation between four (4) consecutive elements of a spherical triangle*



Proof: Let ABC be a spherical triangle. Let four consecutive elements a, C, b, A be taken as a, C, b and A. But, the side b which is between two angles A and C is termed as *inner side* and *a* is other side. Again the Angle C , which is between the sides *a* and *b* is termed as inner angle and the angle A is other angle.

$$We\ have\ \Delta ABC,\ Cosa = Cosb\ Cosc + Sinb\ Sinc\ CosA \qquad (1)$$

$$Cosc = Cosa\ Cosb + SinA\ Sinb\ CosC \qquad (2)$$

And $Sinc = SinC\frac{Sina}{SinA}$ (3)

Putting the values of Cosc and Sinc from (2) and (3) in Eq.(1) we get,

$$Cosa = Cosb\ (Cosa\ Cosb + Sina\ Sinb\ CosC) + Sinb\ Sin\ C\ \frac{Sina}{SinA}\ CosA$$

$$Cosa = Cosa\ Cos^2b + Sina\ Sinb\ Cosb\ CosC + Sina\ Sinb\ SinC\ CotA$$

$$Cosa\ (1 - Cos^2b) = Sina\ SinbCosb\ CosC + Sina\ SinbSinC\ CotA$$

$$Cosa\ Sin^2b = Sina\ SinbCosb\ CosC + Sina\ Sinb\ SinC\ CotA \qquad (4)$$

Dividing by sina sinb on both sides of Eq.(4) we get,

$$\frac{CosaSin^2b}{Sina\ Sinb} = \frac{Sina\ Sinb}{Sina}\frac{Cosb\ CosC}{Sinb} + \frac{Sina\ Sinb}{Sina}\frac{SinC\ CotA}{Sinb}$$

$$Cota\ Sinb = Cosb\ CosC + SinC\ CotA$$

$$Cosb\ CosC = Sinb\ CotA - SinC\ CotA$$

The above formula can be easily remembered as

Cos (Inner Side). Cos (Inner angle)= Sin (Inner side) .Cot (other side) –Sin (Inner angle) Cot (other angle)

[4] **Napier's rule in Circular Parts (formula)** : In a spherical triangle Prove that

[i] $tan\left(\frac{A+B}{2}\right) = \frac{Cos\frac{(a-b)}{2}}{Cos\frac{(a+b)}{2}}\ Cot\ \frac{C}{2}$

[ii] $tan\left(\frac{A-B}{2}\right) = \frac{Sin\frac{(a-b)}{2}}{Sin\frac{(a+b)}{2}}\ Cot\ \frac{C}{2}$

[iii] $tan\left(\frac{a+b}{2}\right) = \frac{Cos\frac{(A-B)}{2}}{Cos\frac{(A+B)}{2}}\tan\frac{c}{2}$

[iv] $tan\left(\frac{a-b}{2}\right) = \frac{Sin\frac{(A-B)}{2}}{Sin\frac{(A+B)}{2}}\tan\frac{c}{2}$

Proof [1]:

By sine formula we have,

$\frac{SinA}{Sina} = \frac{SinB}{Sinb} = \frac{SinC}{Sinc} = k\ (suppose)$ (1)

SinA + SinB= k(Sina+Sinb) (2)

Again by supplemental cosine formula , we have

$CosA = -CosB\ CosC = SinB\ SinCCosa$

$CosB = -CosC\ CosA + SinC\ SinA\ Cosb$

$CosA + CosB = -CosC(CosA + CosB) + SinC(\ SinB\ Cosa + SinC\ Cosa)$

$(CosA + CosB)(1 + CosC) = SinC[SinB\ Cosa + SinC\ Cosb]$

$(CosA + CosB)(1 + CosC) = SinC[k\ Sinb\ Cosa + k\ SincCosb]\ from\ (1)$

$= k\ SinC\sin(a+b)$

$$(CosA + CosB) = k\frac{SinC}{(1 + CosC)}Sin\ (a+b)$$

Or $(CosA + CosB) = k\ \dfrac{\frac{2CosC/2}{2}\,SinC/2}{2Cos^2\ C/2} Sin\ (a + b).$

Or $(CosA + CosB) = k\ tan\dfrac{C}{2}\ Sin(a + b)$ (3)

Now, $tan\left(\frac{A+B}{2}\right) = \dfrac{Sin\left(\frac{A+B}{2}\right)}{Cos\left(\frac{A+B}{2}\right)}.\dfrac{2.Cos\left(\frac{A-B}{2}\right)}{2.Cos\left(\frac{A-B}{2}\right)}$ *(for our convenience)*

Or $tan\left(\frac{A+B}{2}\right) = \dfrac{SinA+SinB}{CosA+CosB}$

Or $tan\left(\frac{A+B}{2}\right) = \dfrac{k(Sina+Sinb)}{k\tan\frac{C}{2}Sin\ (a+b)}$

Or $tan\left(\frac{A+B}{2}\right) = \dfrac{2\ Sin\left(\frac{a+b}{2}\right)}{2\ Sin\left(\frac{a+b}{2}\right)}\ \dfrac{Cos\left(\frac{a-b}{2}\right)}{Cos\left(\frac{a+b}{2}\right)}\ \text{Cot}\ \dfrac{C}{2}$

********END********

Mathematics Major

SEMESTER V

Paper Code: 503

Spherical Trigometry and Astronomy

Marks= (60+15 Internal Assessment)

Unit 1: Section of a sphere by a plane:

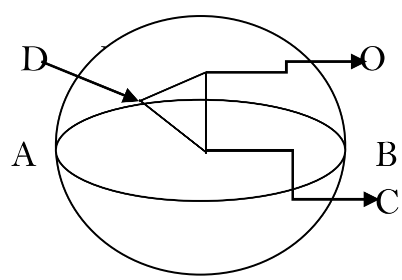


Fig.1

If a plane cuts a sphere, its intersection is a circle. In Fig.1Ab be the section of the surface of a sphere by any plane and O being the centre of the sphere . From O draw OC perpendicular to the plane .Consider any point D on the section AB and join OD and CD. Since OC is perpendicular to CD, a line lying in the plane,

Therefore $\angle OCD = \pi/2$, Now from the right angled ΔOCD we have

$CD^2 = OD^2 - OC^2$; $CD = \sqrt{(OD^2 - OC^2)}$ = a constant.

Thus all points on the plane section are at the same distance from a fixed point C. Thus, the section of a sphere by a plane is a circle whose centre is the fixed point C.

Great and small circle:

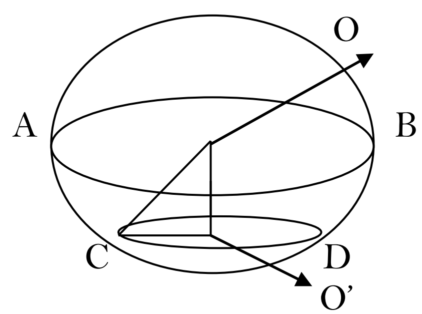


Fig.2

A plane can be made to cut a sphere as to get circles of different sizes. Largest circle is formed when the plane passes through the centre of the sphere. The circle is called 'great circle'. If the plane does not pass through the centre of the sphere, the circle is called 'small circle' which is in Fig.2.

Spherical Triangles:

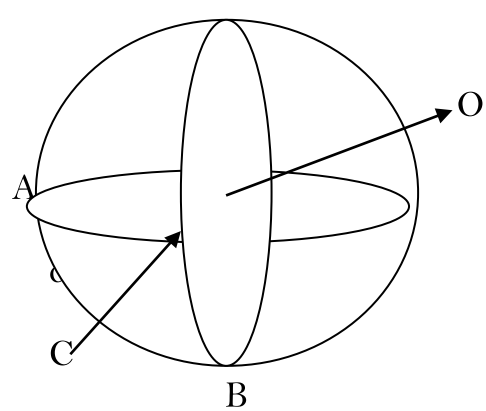


Fig.3

Let A, B, C be the three points on the surface of a sphere. The figure ABC is formed by joining a, B and C by arcs of great circles. Thus, only one spherical triangle can be formed by joining the points A, B and C by arcs of great circles. Now, AB=c, AC=b and BC= a respectively. The sides BC, CA and AB of the spherical triangle is denoted by a, b and c at and is measured by the angle subtended by them at the centre of the sphere. Hence, the three sides a, b and c and the three angles A, B and C are the six (6 elements) of the spherical triangle of Δ ABC.

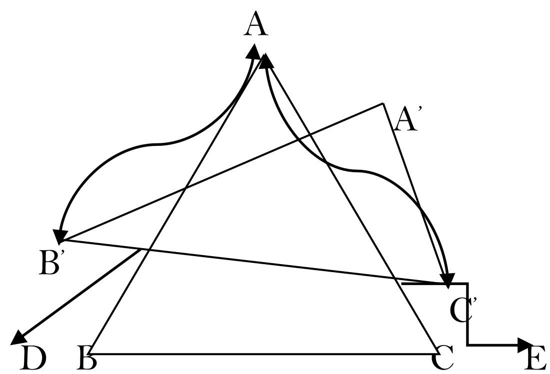


Fig. 4

Polar Triangle and let ABC be any spherical triangle and let A' , B' and C' be those poles of the arcs BC, CA and AB respectively which lie on the same sides of them as the opposite angles a, B and C i.e AA' , BB' and CC' and each < π/2.The triangle A' B'C' formed by joining A' , B' and C' by arcs of great circles is called the polar triangle of the given triangle ABC, which in turn is called the primitive triangle.

Properties of Polar triangle:

Prop. I: *If one triangle of the polar triangle of another, then the latter will be the polar triangle of the former.*

Proof: Let ABC be any spherical triangle and $A'B'C'$ be the polar triangle of ABC. Since B' is the pole of AC, then arc AB' is a quadrant, C' is the pole of AB then AC' is a quadrant ; thus both AB', AC' are quadrants and hence A is a pole of $B'C'$ similarly B is a pole of $C'A'$, C is a pole of $A'B'$ and C, C' lie on the same side of $A'B'$. Therefore ABC is the polar triangle of $A'B'C'$.

Prop. II: *The sides and angles of the polar triangle are respectively the supplement of the angles and sides of the primitive triangle.*

Proof: Let ABC be any spherical triangle and $A'B'C'$ its polar triangle. Let AB and AC meet $B'C'$ at the points D and E respectively. Since A is the pole of $B'C'$.

Now B' is a pole of CA; then $B'E = \pi/2$
C' is the pole of AB ;then $C'D = \pi/2$
$B'E + C'D = \frac{\pi}{2} + \frac{\pi}{2} = \pi$
$B'E + (C'E + ED) = \pi$
$(B'E + C'E) + DE = \pi$

$$B'C' + \angle A = \pi$$

$$B'C' = \pi - A$$

Similarly, $C'A'$=π-B, A' B'=π-C since ABC is the polar triangle of$A'B'C'$ we have, BC=π-A' , CA= =π-B' and AB=π-C'
Thus if a, b, c, A, B, and C be the elements of the spherical triangle and $a', b,' c'$, A' B', C', then $a' = \pi - A,\quad b' = \pi - B \; and\; c' = \pi - C$
And then $A' = \pi - a, B' = \pi - b,\; C' = \pi - c$
From above relations we conclude that sides and angles of a polar triangle are respectively the supplements of the angles and sides of the primitive triangle.

Some properties of Spherical Triangles

[1] *Any two sides of a spherical are together greater than the third sides.*

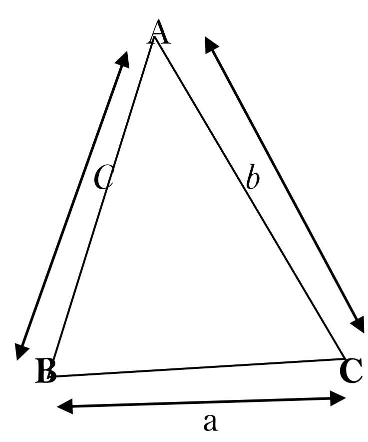


Fig. 5

Let ABC be a spherical triangle then AB, BC and CA are the smaller arcs of great circles joining the points A and B, B and C and C and A respectively. Thus, AB, BC and CA are the shortest distances between the points A and B, B and C and C and A respectively. Therefore, the arc AB being the shortest distance between the points a and

B is less than BC+CA

[2] *The sum of the three sides of a spherical triangle is less than the circumference of a great circle.*

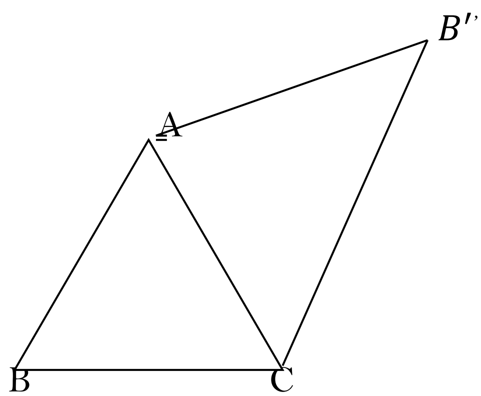


Fig. 6

Let ABC be the spherical triangle. Produce the sides BA and BC to meet at B'.
BA$B' = BCB' = 180^0$ since we know that two great circles bisect each other.
From $\Delta AB'C$ we have, $AB' + CB' > AC$
Now adding BA+BC on the both sides, we get ,

$$(BA + AB') + (BC + CB') > BA + BC + AC$$

or

$$BAB' + BCB' > AB + BC + CA$$

$$or \qquad 180^0 + 180^0 > a + b + c$$

$$\text{Or} \qquad a + b + c < 360^0$$

[3] *The three angles of a spherical triangle are together greater than two-right angles and less than six right angles.*

Let ABC be the spherical triangle and $A'B'C'$
Its polar triangle. If a' $,b'$ $,c'$ be the sides of the polar triangle $A'B'C'$ we have applying properties (ii) $a' + b' + c' < 2\pi$

$$(\pi - A) + (\pi - B) + (\pi - C) < 2\pi$$

$$3\pi - (A + B + C) < 2\pi$$

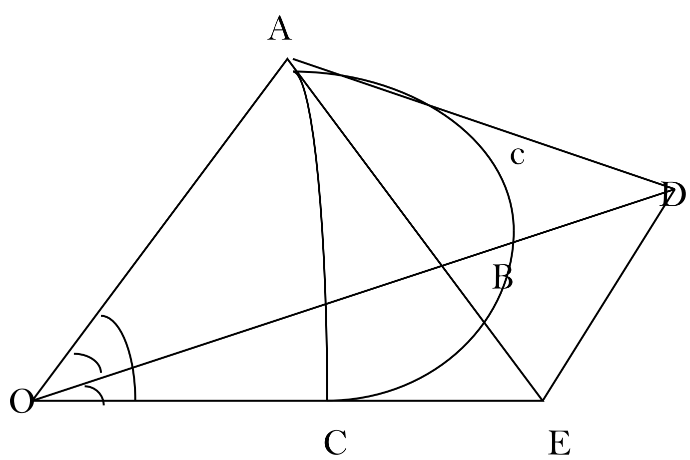
$$A+B+C > \pi$$

So that $\qquad$ A+B+C> $2(\frac{\pi}{2})$ $\qquad$ (1)

Again we know that each angle of a spherical triangle is less than two-right angles i.e. less than π.
Therefore, A+B+C< 3π
A+B+C< $6\,(\frac{\pi}{2})$ $\qquad$ (2)
Now, Equations (1) and (2) prove the required results.

Relation between sides and angles of a spherical triangle

1. **Cosine Formula**



Let ABC be a spherical triangle whose sides AB, BC and CA are arcs of great circles of the sphere whose centre is O.

Let Ad and AE be the tangents at A to the great circle arcs AB and AC. Then radius OA is perpendicular to AD and AE.

$\angle OAD = \angle OAE = \pi/2$

Let Ad meet OB produced at D and AE meet OC produced at E. Join ED.

$\angle OAD = \frac{\pi}{2}; \quad OD^2 = OA^2 + AD^2$ (1)

$\angle OAE = \frac{\pi}{2}\ ; \quad OE^2 = OA^2 + AE^2$ (2)

$Now\ from\ the\ plane\ triangles\ ADE\ and\ ODE\ we\ by\ Cosine\ formula$

$DE^2 = AD^2 + AE^2 - 2AD.Ae\ CosA$ (3)

$DE^2 = OD^2 + OE^2 - 2OD.OE\ Cosa$ (4)

$Equating\ Eqs.(3) and\ (4) we\ have$

$AD^2 + AE^2 - 2AD.AE\ CosA\ = OA^2 + AD^2) + \ (OA^2 + AE^2) - 2OD.OE\ Cosa$

from (1) and (2)

$$OD.OE\ Cosa = OA^2 + AD.AE\ CosA$$

$$Cosa = \frac{OA}{OE}.\frac{OA}{OD} + \frac{AE}{OE}.\frac{Ad}{OD}\ Cos\ A$$

Hence Cosa = Cosb Cosc+ Sinb Sinc CosA

Similarly we can write

Cosb= Cosc Cosa +Sinc Sina Cos B

Cosc = Cosa Cosb + Sina Sinb CosC

The above three formulae can be written as another form such that

$\frac{coa-cosb\ cosc}{sinb\ sinc} = CosA$; $CosB = \frac{cosb-cosa\ cosa}{sina\ sinc}$; $CosC = \frac{cosc-cosa\ cosb}{\sin a \sin b}$

1.1 Supplemental Cosine formula :

Let ABC be the spherical triangle and $A'B'C'$ be its polar triangle . Since the sides and angles of a polar triangle are respectively supplements of the angles and sides of the primitive triangle, therefore

$a' = \pi - A$; $b' = \pi - B$; $c' = \pi - C$

And $A' = \pi - a$; $B' = \pi - b$; $C' = \pi = c$,

Now applying cosine formula on the polar triangle $A'B'C'$ we have

$$cosa' = cosb'cosc' + sinb'sinc' \sin A'$$

Cos(π-A)= Cos(π-B) Cos(π-C) + Sin(π-B)Sin (π-C) cos (π-a)

Or – CosA=-CosB (-CosC)+SinB Sin C (-cosa)

CosA = − CosB CosC+SinB Sin C Cosa

Similarly

Cos B= − CosC CosA + SinC SinA Cosb

CosC= − CosA CosB + SinB Sin A Cosc

The above three formulae can be written as

$$Cosa = \frac{CosA + CosB\ CosC}{Sin\ B\ Sin\ C}; Cosb = \frac{CosB + CosC\ CosA}{SinC\ Sin\ A};$$

$$cosc = \frac{CosC+CosA\ CosB}{Sin\ A\ Sin\ B}$$

+++++++++++++++++++++++++++++++++